Differential equation for Local Magnetization in the Boundary Ising Model

Oleg Miroshnichenko*

Bogolyubov Institute for Theoretical Physics, Kiev 03143, Ukraine

Abstract

We show that the local magnetization in the massive boundary Ising model on the half-plane with boundary magnetic field satisfies second order linear differential equation whose coefficients are expressed through Painleve function of the III kind.

1 Introduction

In the work [1] a very simple and elegant derivation of the famous Painleve equations for the spin-spin correlation function in the scaling Ising model with zero magnetic field was given. The approach used in that work was also applied in [2] to derive finite volume form factors of spin field in the Ising theory and in [3] to derive the differential equation for spin-spin correlation functions in the Ising theory on a pseudosphere. Here we apply this approach to derive differential equation for local magnetization (i. e. one-point correlation function of spin field) in the boundary Ising model on the half-plane with boundary magnetic field. It turns out to be a second order linear differential equation whose coefficients are expressed through Painleve function of the III kind. Supplied with appropriate boundary condition it uniquely defines the local magnetization as a function of the distance to the boundary. Besides being interesting in itself, such a representation for local magnetization may be more convenient for numerical calculation in comparison with conventional form factor expansion, especially in the short distance region.

As is well known [4], there are two essentially different types of conformal boundary conditions (b. c.) in conformal Ising field theory. The so called "free" b. c. corresponds to the universality class represented by the lattice Ising model with unrestricted spins on the boundary. The so called "fixed" b. c. corresponds to the universality class represented by the lattice Ising model with boundary spins all fixed in the same direction ("+" or "−", so there is more precisely two different "fixed" b. c.). All this b. c. correspond to the fixed points of the boundary renormalization group flow. The most general local b. c. in the Ising field theory is the "free" b. c. perturbed by the boundary spin operator (which is the only non-trivial relevant boundary operator in the case of "free" b. c.). This b. c. corresponds to the renormalization group flow from "free" b. c. towards one of the "fixed" b. c. [5]. More generally, one may consider conformal Ising field theory with "free" b. c., perturbed by both boundary spin operator and bulk thermal operator [6]. This theory

*E-mail: oleg2@inbox.ru

describes the continuum limit in the vicinity of the critical point of the lattice Ising model with zero magnetic field in the bulk and with boundary magnetic field being suitably rescaled. Let us briefly list known results about the local magnetisation in this theory defined on the half-plane.

The form factor expansion for local magnetization $\bar{\sigma}(t)$ was written down in [7] using exact expression for boundary state obtained in [6]:

$$\bar{\sigma}(t) = \sigma_0 \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f_k\right) \tag{1}$$

$$f_k = -\frac{1}{\pi^2} \int_0^{\infty} du_1 \ldots \int_0^{\infty} du_k \prod_{l=1}^{k} \frac{\operatorname{ch} u_l - 1}{\operatorname{ch} u_l + \operatorname{ch} u_{l+1}} \left(\frac{\operatorname{ch} u_l + 1 - \lambda}{\operatorname{ch} u_l - 1 + \lambda}\right) e^{-t \operatorname{ch} u_l} \tag{2}$$

$$t = 2m\mathrm{y}, \qquad \lambda = \frac{4\pi h^2}{m}, \qquad \sigma_0 = 2^{\frac{1}{12}} e^{-\frac{1}{8}} A^{\frac{3}{2}} m^{\frac{1}{8}} \tag{3}$$

where $m \sim T - T_c$ is the mass of a particle, h - scaling boundary magnetic field, y - distance from the boundary, σ_0 - magnetization on the infinite plane, $A = 1.28243\ldots$ is Glaisher's constant. Here and later on we always consider the low temperature phase $T < T_c$, unless it is specially pointed out. It is also implied conformal normalization of spin field:

$$|x - x'|^{\frac{1}{4}} \langle \sigma(x) \sigma(x') \rangle \to 1, \quad \text{as } x \to x' \tag{4}$$

With this normalization $\bar{\sigma}(t, \lambda) \to \sigma_0$ as $t \to \infty$. The expansion (1)-(3) was first obtained in [9] from lattice model calculations. It was also shown in [8],[9] that in the cases of "free" ($h = 0$) and "fixed" ($h \to \pm\infty$) b. c. local magnetization can be expressed through Painleve function of the III kind:

$$\bar{\sigma}_{free}(t) = \sigma_0 \exp\left\{\frac{1}{4}\varphi(t) + \frac{1}{4}\int_t^{\infty} \left[e^{-\varphi(r)} - 1 + \frac{r}{2}\left(\operatorname{sh}^2 \varphi(r) - (\varphi'(r))^2\right)\right] dr\right\} \tag{5}$$

$$\bar{\sigma}_{fixed}(t) = \sigma_0 \exp\left\{-\frac{1}{4}\varphi(t) + \frac{1}{4}\int_t^{\infty} \left[1 - e^{\varphi(r)} + \frac{r}{2}\left(\operatorname{sh}^2 \varphi(r) - (\varphi'(r))^2\right)\right] dr\right\} \tag{6}$$

where $\varphi(r)$ is the solution of radial sinh-Gordon equation:

$$\varphi'' + \frac{1}{r}\varphi' = \frac{1}{2}\operatorname{sh} 2\varphi \tag{7}$$

satisfying asymptotic conditions:

$$\varphi(r) = -\ln\left(-\frac{1}{2}r\Omega\right) + O\left(r^4\Omega^2\right), \quad \text{as } r \to 0, \quad \Omega = \ln\left(\frac{e^{\gamma}}{8}r\right) \tag{8}$$

$$\varphi(r) = \frac{2}{\pi}K_0(r) + O\left(e^{-3r}\right), \quad \text{as } r \to \infty \tag{9}$$

where γ is the Euler's constant, $K_0(x)$ is the modified Bessel function of zeroth order. As is known, $\varphi(x)$ is related to Painleve function of the III kind $\eta(x)$ as $\eta(x) = e^{-\varphi(2x)}$. More about this function see [10],[11].

In the case when bulk is critical ($m = 0$) it was shown in [12] that:

$$\bar{\sigma}(\mathrm{y}) = h 2^{\frac{5}{4}} \pi^{\frac{1}{2}} (2\mathrm{y})^{\frac{3}{8}} \Psi\left(1/2, 1, 8\pi h^2 \mathrm{y}\right) \tag{10}$$

where

$$\Psi\left(a,c,x\right)=\frac{1}{\Gamma\left(a\right)}\int_{0}^{\infty}e^{-xt}t^{a-1}\left(1+t\right)^{c-a-1}dt \tag{11}$$

is a solution of degenerate hypergeometric equation.

The qualitative behavior of $\bar{\sigma}\left(t,\lambda\right)$ is well understood [7]. On the whole interval $(0,\infty)$ $\bar{\sigma}_{free}\left(t\right)$ monotonically increases and $\bar{\sigma}_{fixed}\left(t\right)$ monotonically decreases, both approaching σ_0 as $t\to\infty$. For small t:[1]

$$\bar{\sigma}_{free}\left(t\right)\sim t^{\frac{3}{8}} \tag{12}$$

$$\bar{\sigma}_{fixed}\left(t\right)\sim t^{-\frac{1}{8}} \tag{13}$$

For $0<\lambda<2$ $\bar{\sigma}\left(t,\lambda\right)$ remains monotonically increasing. Its values near the boundary are somewhat enhanced by the presence of boundary magnetic field, the leading term of its short distance asymptotic become dressed by logarithm:

$$\bar{\sigma}\left(t\right)\sim t^{\frac{3}{8}}\ln t \tag{14}$$

For $\lambda>2$ it possesses a maximum in some point. As $\lambda\to\infty$ this maximum turns into a very sharp peak located in the region $t\sim\lambda^{-1}$ near the boundary, its shape being described by (10), (11). For $t\ll\lambda^{-1}$ $\bar{\sigma}\left(t,\lambda\right)$ behaves as (14), while for $t\gg\lambda^{-1}$ its behavior coincides with one under "fixed" b. c. (6). This dependence reflects the renormalization group cross-over between "free" and "fixed" b. c.

The main result of this paper is that for arbitrary λ:

$$\bar{\sigma}\left(t,\lambda\right)=u\left(t,\lambda\right)\bar{\sigma}_{free}\left(t\right) \tag{15}$$

where $\bar{\sigma}_{free}\left(t\right)$ is given by (5), and $u\left(t,\lambda\right)$ is the solution of differential equation:

$$u''-\left(\varphi'-\operatorname{ch}\varphi+\lambda\right)u'+\frac{1}{2}\lambda\left(\varphi'-\operatorname{ch}\varphi+1\right)u=0 \tag{16}$$

satisfying asymptotic condition:

$$u\left(t\right)=1+O\left(t^{-\frac{1}{2}}e^{-t}\right),\quad \text{as } t\to\infty \tag{17}$$

(Here $\varphi\left(t\right)$ is the same function as in (5), (6) and the strokes stand for detivatives with respect to t.)

Let us make some remarks on the equation (16). One can see that when $\lambda=0$, the only solution of (16) satisfying (17) is $u\left(t\right)=1$. When $\lambda\to\infty$ (16) turns into a first order differential equation which upon integrating and fixing integration constant with the help of (17) yields (6). In the massless limit ($t\to0$, $\lambda\to\infty$, $t\lambda$ kept fixed) (16) turns into a degenerate hypergeometric equation. Its solution can be fixed by "sewing" its asymptotic as $t\lambda\to\infty$ with asymptotic of (6) as $t\to0$, and this yields (10), (11). The fact that in massless limit we reproduce the result of [12] is not very surprising because the approach we used to derive (15), (16) is a generalization of one used in [12]. Concerning the relation between the form factor expansion (1), (2), (3) and our result (15), (16) we just note that it seems to be very difficult to show directly that (1), (2) satisfy (15), (16). In any case, it is beyond the analytic abilities of the author.

[1] Note that comparing the coefficient in the short distance asymptotic of $\sigma_{fixed}\left(t\right)$, that follows from (6), with the result $\sigma_{fixed}\left(\mathrm{y}\right)=2^{\frac{1}{4}}\left(2\mathrm{y}\right)^{-\frac{1}{8}}$ (for $m=0$) of [13], one obtains the identity $\int_0^{\infty}\left(1-e^{-\varphi(r)}\right)dr=\ln 2$

Being second order linear differential equation, (16) possesses two linearly independent solutions. Their asymptotics as $t \to \infty$ are $u_1(t) \sim 1$ and $u_2(t) \sim e^{(\lambda-1)t}$. Hence, for $\lambda > 1$ the condition that $u(t) \to 1$ as $t \to \infty$ is sufficient to fix the solution uniquely. For $\lambda \leq 1$ more strict condition (17) is required, which follows from form factor expansion (1), (2). Another linearly independent solution in this case also has physical meaning. As explained in [6], for $\lambda < 1$ there exists metastable state characterized by asymptotic behavior $\bar{\sigma}(t) \to -\sigma_0$ as $t \to \infty$ and corresponding to the boundary bound state in the hamiltonian picture with "space" being half-line and "time" axis being parallel to the boundary. The local magnetization $\bar{\sigma}_1(t,\lambda)$ in this state can be obtained from $\bar{\sigma}(t,\lambda)$ by analytic continuation $h \to -h$. Clearly, it is also a solution of (16). As it was shown in [14] its asymptotic as $t \to \infty$ is:

$$\bar{\sigma}_1(t,\lambda) = -\sigma_0 + \sigma_0 \left(\frac{\lambda}{2-\lambda}\right)^{\frac{1}{2}} e^{-(1-\lambda)t} + \frac{\sigma_0}{4\sqrt{2\pi}}\left(\frac{2}{\lambda} - 1\right) t^{-\frac{3}{2}} e^{-t} + o\left(t^{-\frac{3}{2}} e^{-t}\right) \tag{18}$$

The presence of exponential term $\sim e^{-(1-\lambda)t}$ in (18) agrees with (16).

In the rest of the paper we present the details of our derivation of (15), (16).

2 Ising field theory in the bulk

In this section we briefly recall some well known facts [1] about the structure of the Ising field theory in the bulk needed for further computations. As is known the Ising field theory in zero magnetic field is equivalent to the free Majorana fermion theory with euclidean action:

$$S = \frac{1}{2\pi}\int \left(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} - im\bar{\psi}\psi\right) d^2x \tag{19}$$

Here we have assumed that the theory is defined on an infinite plane $\mathbb{R}^2$, whose points x are labelled by cartesian coordinates $(\mathrm{x},\,\mathrm{y}) = (\mathrm{x}(x)\,,\,\mathrm{y}(x))$, and $d^2x \equiv d\mathrm{x}\,d\mathrm{y}$. Complex coordinates are defined as $z(x) = \mathrm{x} + i\mathrm{y}$, $\bar{z}(x) = \mathrm{x} + i\mathrm{y}$, and the derivatives $\partial, \bar{\partial}$ in (19) stand for $\partial_z = \frac{1}{2}(\partial_\mathrm{x} - i\partial_\mathrm{y})$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_\mathrm{x} + i\partial_\mathrm{y})$ respectively. The ciral components ψ, $\bar{\psi}$ of fermi field satisfy Dirac's equations:

$$\bar{\partial}\psi = -\frac{im}{2}\bar{\psi}, \qquad \partial\bar{\psi} = \frac{im}{2}\psi \tag{20}$$

Their normalization in the action (19) corresponds to the following short-distance limit of the operator products

$$z\psi(x)\,\psi(0) \to 1, \qquad \bar{z}\bar{\psi}(x)\,\bar{\psi}(0) \to 1, \qquad \text{as } x \to 0 \tag{21}$$

The order $\sigma(x)$ and disorder $\mu(x)$ fields are semi-local with respect to the fermi fields; the products

$$\psi(x)\,\sigma(0)\,, \qquad \psi(x)\,\mu(0)\,, \qquad \bar{\psi}(x)\,\sigma(0)\,, \qquad \bar{\psi}(x)\,\mu(0) \tag{22}$$

acquire a minus sign when the point x is taken around zero point. The fields $\psi(x)$ and $\bar{\psi}(x)$ in the products (22) can be expanded in the complete set of solutions of Dirac's equations (20) having this monodromy property:

$$\begin{pmatrix} \psi(x) \\ \bar{\psi}(x) \end{pmatrix} = \sum_{n\in\mathbb{Z}} a_n \begin{pmatrix} u_{-n}(x) \\ \bar{u}_{-n}(x) \end{pmatrix} + \bar{a}_n \begin{pmatrix} v_{-n}(x) \\ \bar{v}_{-n}(x) \end{pmatrix} \tag{23}$$

where

$$\left(\begin{array}{c} u_n(x) \\ \bar{u}_n(x) \end{array}\right) = \left(\frac{m}{2}\right)^{\frac{1}{2}-n} \Gamma\left(n+\frac{1}{2}\right) \left(\begin{array}{c} e^{i\left(n-\frac{1}{2}\right)\theta} I_{n-\frac{1}{2}}(mr) \\ -ie^{i\left(n+\frac{1}{2}\right)\theta} I_{n+\frac{1}{2}}(mr) \end{array}\right) \tag{24}$$

$$\left(\begin{array}{c} v_n(x) \\ \bar{v}_n(x) \end{array}\right) = \left(\frac{m}{2}\right)^{\frac{1}{2}-n} \Gamma\left(n+\frac{1}{2}\right) \left(\begin{array}{c} ie^{-i\left(n+\frac{1}{2}\right)\theta} I_{n+\frac{1}{2}}(mr) \\ e^{-i\left(n-\frac{1}{2}\right)\theta} I_{n-\frac{1}{2}}(mr) \end{array}\right) \tag{25}$$

(here r, θ are polar coordinates, i. e. $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$ and I_ν are modified Bessel functions). The coefficients a_n, $\bar{a}_n$ in (23) are understood as operators acting on the space of fields.

It can be easily shown that for any two solutions $\Psi_1 = \left(\psi_1(x), \bar{\psi}_1(x)\right)$, $\Psi_2 = \left(\psi_2(x), \bar{\psi}_2(x)\right)$ of Dirac's equations which change sign after the point x is taken around zero the integral

$$(\Psi_1, \Psi_2) = \frac{1}{2\pi i} \oint_{C_0} \psi_1(x)\, \psi_2(x)\, dz - \bar{\psi}_1(x)\, \bar{\psi}_2(x)\, d\bar{z} \tag{26}$$

over a contour C_0 encircling zero (in counter-clockwise direction) does not change under continuous deformation of C_0 and therefore defines a bilinear form on the space of such solutions. The solutions $U_n = (u_n, \bar{u}_n)$ and $V_n = (v_n, \bar{v}_n)$ satisfy the following orthogonality properties with respect to this bilinear form:

$$(U_n, U_m) = \delta_{n+m,0}\ , \qquad (V_n, V_m) = \delta_{n+m,0}\ , \qquad (U_n, V_m) = 0 \tag{27}$$

Let us also write down the following differentiation formulas which are useful in computations with U_n and V_n:

$$\begin{aligned} \partial U_n &= \left(n-\frac{1}{2}\right) U_{n-1}\ , \qquad \bar{\partial} U_n = \frac{m^2}{2(2n+1)} U_{n+1} \\ \partial V_n &= \frac{m^2}{2(2n+1)} V_{n+1}\ , \qquad \bar{\partial} V_n = \left(n-\frac{1}{2}\right) V_{n-1} \end{aligned} \tag{28}$$

(here we denote $\partial U_n \equiv (\partial u_n(x), \partial \bar{u}_n(x))$, etc.).

Using relations (27) one can express operators a_n, $\bar{a}_n$ in terms of contour integrals:

$$a_n = \frac{1}{2\pi i} \oint_{C_0} u_n(x)\, \psi(x)\, dz - \bar{u}_n(x)\, \bar{\psi}(x)\, d\bar{z} \tag{29}$$

$$\bar{a}_n = \frac{1}{2\pi i} \oint_{C_0} v_n(x)\, \psi(x)\, dz - \bar{v}_n(x)\, \bar{\psi}(x)\, d\bar{z} \tag{30}$$

This representation can be used to show that they satisfy canonical commutation relations:

$$\{a_n, a_m\} = \delta_{n+m,0}\ , \qquad \{\bar{a}_n, \bar{a}_m\} = \delta_{n+m,0}\ , \qquad \{a_n, \bar{a}_m\} = 0 \tag{31}$$

The fields σ and μ are "primary" with respect to the algebra (31), i. e. they satisfy relations:

$$a_n \sigma = 0\ , \qquad \bar{a}_n \sigma = 0\ , \qquad a_n \mu = 0\ , \qquad \bar{a}_n \mu = 0 \tag{32}$$

for $n>0$, as well as

$$\begin{aligned}
a_0\sigma &= \frac{\omega}{\sqrt{2}}\mu\ , \qquad a_0\mu=\frac{\bar{\omega}}{\sqrt{2}}\sigma \\
\bar{a}_0\sigma &= \frac{\bar{\omega}}{\sqrt{2}}\mu\ , \qquad \bar{a}_0\mu=\frac{\omega}{\sqrt{2}}\sigma
\end{aligned} \tag{33}$$

where $\omega=e^{i\pi/4}$ and $\bar{\omega}=e^{-i\pi/4}$. These equations define the fields σ and μ up to normalization. In what follows we will assume conformal normalization of fields σ and μ:

$$|x|^{\frac{1}{4}}\,\sigma\left(x\right)\sigma\left(0\right)\to 1\ , \qquad |x|^{\frac{1}{4}}\,\mu\left(x\right)\mu\left(0\right)\to 1\ , \qquad \text{as } x\to 0 \tag{34}$$

As it is shown in [1], first and second order descendants of σ and μ with respect to the algebra a_n, $\bar{a}_n$ are expressed in terms of coordinate derivatives of σ and μ:

$$\begin{aligned}
a_{-1}\sigma &= \frac{\omega}{\sqrt{2}}4\partial\mu\ , \qquad a_{-1}\mu=\frac{\bar{\omega}}{\sqrt{2}}4\partial\sigma \\
\bar{a}_{-1}\sigma &= \frac{\bar{\omega}}{\sqrt{2}}4\bar{\partial}\mu\ , \qquad \bar{a}_{-1}\mu=\frac{\omega}{\sqrt{2}}4\bar{\partial}\sigma
\end{aligned} \tag{35}$$

$$\begin{aligned}
a_{-2}\sigma &= \frac{\omega}{\sqrt{2}}\frac{8}{3}\partial^2\mu\ , \qquad a_{-2}\mu=\frac{\bar{\omega}}{\sqrt{2}}\frac{8}{3}\partial^2\sigma \\
\bar{a}_{-2}\sigma &= \frac{\bar{\omega}}{\sqrt{2}}\frac{8}{3}\bar{\partial}^2\mu\ , \qquad \bar{a}_{-2}\mu=\frac{\omega}{\sqrt{2}}\frac{8}{3}\bar{\partial}^2\sigma
\end{aligned} \tag{36}$$

This observation is very important for the method of [1] to work.

The Majorana theory (19) corresponds to both high and low temperature phases of the Ising model in the vicinity of its critical point T_c depending of the choice of the sign of the mass parameter m in (19). Our definition in (33) corresponds to the identification of the case $m>0$ with the ordered phase $T<T_c$, while the case $m<0$ is identified with the disordered phase $T>T_c$. From now on we will consider the ordered phase $m>0$.

3 "Free" and "fixed" boundary conditions

In this section we consider the Ising field theory defined on the half-plane $\mathrm{y}>0$. As a warm-up exercise let us first rederive formulas (5), (6) for local magnetization in the cases of "free" and "fixed" b. c. Explicit expressions for boundary states for "free" and "fixed" b. c. was obtained in [4]. It follows from this expressions that the fields ψ and $\bar{\psi}$ satisfy the following b. c.:

$$\left(\psi-\bar{\psi}\right)\big|_{\mathrm{y}=0} = 0 \qquad \text{(for "free" b. c.)} \tag{37}$$

$$\left(\psi+\bar{\psi}\right)\big|_{\mathrm{y}=0} = 0 \qquad \text{(for "fixed" b. c.)} \tag{38}$$

Suppose that $\left(\chi\left(x\right),\bar{\chi}\left(x\right)\right)$ is a double-valued solution of Dirac's equations defined on the half-plane $\mathrm{y}>0$ (with punctured point x_0) such that it changes sign when x is taken around x_0, decays sufficiently fast as $|x|\to\infty$, and satisfies b. c.:

$$\left(\chi-\bar{\chi}\right)\big|_{\mathrm{y}=0} = 0 \qquad \text{(for "free" b. c.)} \tag{39}$$

$$\left(\chi+\bar{\chi}\right)\big|_{\mathrm{y}=0} = 0 \qquad \text{(for "fixed" b. c.)} \tag{40}$$

Then the following identity holds:

$$\langle\, (\oint_{C_{x_0}} \chi(x)\,\psi(x)\,dz - \bar{\chi}(x)\,\bar{\psi}(x)\,d\bar{z}\,)\,\mu(x_0)\,\rangle = 0 \tag{41}$$

where C_{x_0} is a contour encircling the point x_0. This is because the integral on the left-hand side of (41) does not changes under continuous deformations of the contour C_{x_0} and therefore one can deform it in such a way that it constitutes of two parts: C_∞ which tends to infinity and C_b which passes along the boundary. Then the integral along C_∞ is zero because $\chi(x)$ and $\bar{\chi}(x)$ decay at infinity and the integral along C_b is zero due to (39) or (40). On the other hand one can shrink C_{x_0} to a small circle around the point x_0 and express the left-hand side of (41) in terms of descendents of μ using the operator product expansions of $\psi(x)$, $\bar{\psi}(x)$ with $\mu(x_0)$. If the singularity of $(\chi(x), \bar{\chi}(x))$ at the point x_0 is not too strong only descendents of not higher than the second order appear in this expression. Since they are expressed in terms of coordinate derivatives of σ this will lead to a differential equation for local magnetization $\langle\sigma(x)\rangle$. The only problem is to find a solution of Dirac's equations $(\chi(x), \bar{\chi}(x))$ satisfying the above conditions. For this purpose we will use the following trick. We will search for this solution in the form of the linear combination:

$$\begin{aligned}\begin{pmatrix}\chi(x)\\ \bar{\chi}(x)\end{pmatrix} &= c_1\begin{pmatrix}\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\end{pmatrix} + c_2\partial\begin{pmatrix}\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\end{pmatrix} +\\ &+ c_3\bar{\partial}\begin{pmatrix}\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\end{pmatrix} + c_4\begin{pmatrix}\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\end{pmatrix} +\\ &+ c_5\partial\begin{pmatrix}\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\end{pmatrix} + c_6\bar{\partial}\begin{pmatrix}\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\end{pmatrix}\end{aligned} \tag{42}$$

where $\langle\ldots\rangle_0$ denotes a correlation function in the Ising field theory defined on the plane and P denotes the reflection in the line $\mathrm{y} = 0$ (i. e. $P(\mathrm{x}, \mathrm{y}) = (\mathrm{x}, -\mathrm{y})$). Obviously each term in (42) is non-zero and as a function of x is a solution of Dirac's equations, change sign when x is taken around x_0 and decay at infinity. The coefficients in this linear combination can be determined from the requirement that it satisfies (39) or (40). Note that we do not need to know the functions $\chi(x)$ and $\bar{\chi}(x)$ in explicit form. What we really need is several terms of their short-distance asymptotics as $x \to x_0$, but the latter can be expressed in terms of the two-point functions $\langle\sigma(x_0)\,\sigma(Px_0)\rangle_0 \equiv G(2m\mathrm{y}_0)$ and $\langle\mu(x_0)\,\mu(Px_0)\rangle_0 \equiv \tilde{G}(2m\mathrm{y}_0)$. As is well known [10] (see also [1]) this functions can be expressed in terms of Painleve function of the III kind as follows:

$$G(t) = \sigma_0\,\mathrm{ch}\left(\frac{1}{2}\varphi(t)\right)\exp\left[\frac{1}{4}\int_t^\infty r\left(\mathrm{sh}^2\,\varphi(r) - (\varphi'(r))^2\right)dr\right] \tag{43}$$

$$\tilde{G}(t) = \sigma_0\,\mathrm{sh}\left(\frac{1}{2}\varphi(t)\right)\exp\left[\frac{1}{4}\int_t^\infty r\left(\mathrm{sh}^2\,\varphi(r) - (\varphi'(r))^2\right)dr\right] \tag{44}$$

where $\varphi(t)$ is the same function as in (5), (6).

Under parity transformation P fermi fields transform as:

$$P\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix} = \begin{pmatrix}-i\bar{\psi}\\ i\psi\end{pmatrix} \tag{45}$$

We have therefore the following identities which follows from the invariance of correlation functions under parity transformation:

$$\left\langle \bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\right\rangle_0\Big|_{\mathrm{y}=0} = i\left\langle \psi(x)\,\mu(x_0)\,\sigma(Px_0)\right\rangle_0\Big|_{\mathrm{y}=0} \tag{46}$$

$$\left\langle \bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\right\rangle_0\Big|_{\mathrm{y}=0} = i\left\langle \psi(x)\,\sigma(x_0)\,\mu(Px_0)\right\rangle_0\Big|_{\mathrm{y}=0} \tag{47}$$

$$\left\langle \bar{\partial}\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\right\rangle_0\Big|_{\mathrm{y}=0} = i\left\langle \partial\psi(x)\,\mu(x_0)\,\sigma(Px_0)\right\rangle_0\Big|_{\mathrm{y}=0} \tag{48}$$

$$\left\langle \bar{\partial}\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\right\rangle_0\Big|_{\mathrm{y}=0} = i\left\langle \partial\psi(x)\,\sigma(x_0)\,\mu(Px_0)\right\rangle_0\Big|_{\mathrm{y}=0} \tag{49}$$

It follows from this identities and Dirac's equations that from all correlation functions that present in the expression (42) only four are linearly independent functions of x on the line $\mathrm{y}=0$ (for example $\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle$, $\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle$, $\langle\partial\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle$, and $\langle\partial\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle$). Hence requiring that (42) satisfy (39) or (40) one obtains four linear constraints for six coefficients $c_1,\ldots c_6$. It turns out that they have non-zero solutions. One of the solutions corresponds to the following linear combination (it does not matter what of the solutions to choose):

$$\begin{aligned}\begin{pmatrix}\chi(x)\\ \bar{\chi}(x)\end{pmatrix} &= \frac{m}{2}\begin{pmatrix}\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\end{pmatrix} - i\partial\begin{pmatrix}\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\end{pmatrix} + \\ &+ i\frac{m}{2}\begin{pmatrix}\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\end{pmatrix} - \bar{\partial}\begin{pmatrix}\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\end{pmatrix},\end{aligned} \tag{50}$$

for "free" b. c., and

$$\begin{aligned}\begin{pmatrix}\chi(x)\\ \bar{\chi}(x)\end{pmatrix} &= i\frac{m}{2}\begin{pmatrix}\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\end{pmatrix} - \partial\begin{pmatrix}\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\end{pmatrix} + \\ &+ \frac{m}{2}\begin{pmatrix}\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\end{pmatrix} - i\bar{\partial}\begin{pmatrix}\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\end{pmatrix},\end{aligned} \tag{51}$$

for "fixed" b. c.

It is now straightforward but somewhat tedious exercise to substitute (50) and (51) in (41) and evaluate the left-hand side. One has to expand ψ and $\bar{\psi}$ using (23), than to evaluate contour integrals using (27), (28) and than to act by the operators a_n, $\bar{a}_n$ on σ and μ using (32), (33) and (35). Due to (32) all terms with descendants of higher than the first order vanish. Finally, taking into account that $\langle\sigma(x_0)\rangle \equiv \bar{\sigma}(2m\mathrm{y}_0)$ depends only on y_0 due to translation invariance, one obtains the following differential equations:

$$2\left(G-\tilde{G}\right)\bar{\sigma}'_{free} - \left(G'-\tilde{G}'+\tilde{G}\right)\bar{\sigma}_{free} = 0 \tag{52}$$

$$2\left(G+\tilde{G}\right)\bar{\sigma}'_{fixed} - \left(G'+\tilde{G}'+\tilde{G}\right)\bar{\sigma}_{fixed} = 0 \tag{53}$$

(the stroke denotes derivative with respect to $t=2m\mathrm{y}_0$). Integrating this equations, substituting (43), (44) and fixing integration constants with the help of asymptotic condition $\bar{\sigma}(t)\to\sigma_0$ as $t\to\infty$ one obtains (5) and (6).

Let us now consider the high temperature phase $T>T_c$. The differential equations in this case can be obtained from (52), (53) by substitution $m\to -m$, $G\rightleftarrows\tilde{G}$:

$$2\left(G-\tilde{G}\right)\bar{\sigma}'_{free} - \left(G'-\tilde{G}'+G\right)\bar{\sigma}_{free} = 0 \tag{54}$$

$$2\left(G+\tilde{G}\right)\bar{\sigma}'_{fixed}-\left(G'+\tilde{G}'-G\right)\bar{\sigma}_{fixed}=0 \tag{55}$$

The only solution of (54) that does not grow exponentially as $t\to\infty$ is $\bar{\sigma}_{free}=0$, while from (55) we obtain:

$$\bar{\sigma}_{fixed,\,T>T_c}=e^{-\frac{1}{2}t}\bar{\sigma}_{fixed,\,T<T_c} \tag{56}$$

in agreement with [9]. This confirms our identification of the case $m>0$ with the low temperature phase. Had we chosen the other choice, we would obtain the exponentially growing solution for $\bar{\sigma}_{fixed}$ in the high temperature phase.

4 Boundary magnetic field

Let us now consider the general case of "free" b. c. perturbed by boundary spin operator σ_B. The latter is identified with degenerate primary boundary field with dimension $\Delta=1/2$ [4]. It can be written in terms of fermion fields as follows [6]:

$$\sigma_B\left(\mathrm{x}\right)=ia\left(\mathrm{x}\right)\left(\psi\left(x\right)+\bar{\psi}\left(x\right)\right)\Big|_{\mathrm{y}=0} \tag{57}$$

where $a\left(\mathrm{x}\right)$ is additional fermionic degree of freedom with two-point function

$$\left\langle a\left(\mathrm{x}\right)a\left(\mathrm{x'}\right)\right\rangle_{free}=\frac{1}{2}\,\mathrm{sign}\left(\mathrm{x}-\mathrm{x'}\right) \tag{58}$$

The action of the theory has therefore the following form:

$$S=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}d\mathrm{x}\int\limits_{0}^{\infty}d\mathrm{y}\left(\psi\bar{\partial}\psi+\bar{\psi}\partial\bar{\psi}-im\bar{\psi}\psi\right)+$$
$$+\int\limits_{-\infty}^{\infty}\left(-\frac{i}{4\pi}\left(\psi\bar{\psi}\right)\Big|_{\mathrm{y}=0}+\frac{1}{2}a\partial_{\mathrm{x}}a\right)d\mathrm{x}+ih\int\limits_{-\infty}^{\infty}a\left(\mathrm{x}\right)\left(\psi+\bar{\psi}\right)\Big|_{\mathrm{y}=0}d\mathrm{x} \tag{59}$$

It leads to the following b. c. for fermion fields [6]:

$$\frac{\partial}{\partial\mathrm{x}}\left(\psi-\bar{\psi}\right)\Big|_{\mathrm{y}=0}=-im\lambda\left(\psi+\bar{\psi}\right)\Big|_{\mathrm{y}=0} \tag{60}$$

where $\lambda=4\pi h^2/m$. We can now proceed in the same way as in the previous section but now instead of (39) or (40) we should require the functions χ and $\bar{\chi}$ to satisfy the condition:

$$\frac{\partial}{\partial\mathrm{x}}\left(\chi-\bar{\chi}\right)\big|_{\mathrm{y}=0}=-im\lambda\left(\chi+\bar{\chi}\right)\big|_{\mathrm{y}=0} \tag{61}$$

in order to write down the Ward identity (41). It turns out that in this case it is necessary to include also terms with second order derivatives in the linear combination (42) in order

to satisfy (61). As a result one obtains the following linear combination:

$$\begin{aligned}\left(\begin{array}{c}\chi(x)\\ \bar{\chi}(x)\end{array}\right) &= i\left(\frac{m}{2}\right)^2(1-2\lambda)\left(\begin{array}{c}\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\end{array}\right)-\\ &-\frac{m}{2}(1-2\lambda)\,\partial\left(\begin{array}{c}\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\end{array}\right)-\frac{m}{2}\bar{\partial}\left(\begin{array}{c}\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\end{array}\right)+\\ &+i\partial^2\left(\begin{array}{c}\langle\psi(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\sigma(x_0)\,\mu(Px_0)\rangle_0\end{array}\right)+\left(\frac{m}{2}\right)^2(1-2\lambda)\left(\begin{array}{c}\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\end{array}\right)-\\ &-i\frac{m}{2}\partial\left(\begin{array}{c}\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\end{array}\right)-i\frac{m}{2}(1-2\lambda)\,\bar{\partial}\left(\begin{array}{c}\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\end{array}\right)+\\ &+\bar{\partial}^2\left(\begin{array}{c}\langle\psi(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\\ \langle\bar{\psi}(x)\,\mu(x_0)\,\sigma(Px_0)\rangle_0\end{array}\right)\end{aligned}\tag{62}$$

Substituting it in (41) and evaluating the left-hand side we obtain the following differential equation for local magnetization $\bar{\sigma}(t)$:

$$\begin{aligned}&\left(G+\tilde{G}\right)\bar{\sigma}''-\left[G'+\tilde{G}'-G+\lambda\left(G+\tilde{G}\right)\right]\bar{\sigma}'+\\ &\quad+\frac{1}{4}\left[G''+\tilde{G}''-\frac{1}{t}\left(G'+\tilde{G}'\right)-2G'-\tilde{G}+2\lambda\left(G'+\tilde{G}'+\tilde{G}\right)\right]\bar{\sigma}=0\end{aligned}\tag{63}$$

It can be brought to a simpler form (16) by means of substitution (15).

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